On a number line, numbers are represented as points and distances. Number lines are useful because they provide a linear representation of all numbers, in order of size. They can represent whole numbers, negative numbers, fractions and decimals and irrational numbers, all on one diagram. They are also useful to model some number computations, especially for addition and subtraction.

Scales on instruments such as thermometers are examples of number lines, so reading a number line has practical use. Graph axes are also number lines, and so understanding number lines is important for more advanced mathematics, not just as a tool for teaching.

Counting Lines

The first way that children encounter “number lines” is as a line or row of numbers or labelled objects for counting. As we shall see below, this is NOT really a number line, even though it is very useful and the numbers are in order. There are many examples of “counting lines”. A school may have a winding path or a row of numbered stepping stones, for children to count as they walk along. There may be a ladder with numbered rungs stretching from the floor to the ceiling. Children may have rows of numbered blocks on their desks. All of these “counting lines” are excellent to support learning to count and learning to add and subtract.

In the diagram below, there is an example of a counting line made of numbered blocks. Note that the first block is labelled with 1 which indicates that we are counting the objects. We can use this ordered sequence of numbered blocks, or even just a row of counting numbers to help with learning to count and for some calculations. Students can see how to write numerals; they can use it as support when they forget which number comes next; they can use it when learning to count backwards; they can show how to count by twos and threes, etc.
Students can also use “counting lines” to illustrate addition and subtraction, and later repeated addition and repeated subtraction as introductory multiplication and division. For example, a child might use the counting line below, made from blue and green coloured blocks, to illustrate or calculate $9 + 5 = 14$. Starting at 9, the student can move 5 more, to get to 14. Another child who is further from attaining fluency with this number fact will still need to count on by ones and this child would draw a different diagram to represent different actions. Similar diagrams and actions can assist children with subtractions.

Moving on to number lines

The counting lines above are very useful, but they are NOT number lines. Making a careful transition from counting lines to number lines is important, because it is the source of many student errors in using number lines.

Number lines are not based on counting, but on measuring from a fixed point. The fixed point, or origin, is 0. The number that labels every point is the distance from the origin (0) to that point. Note that counting lines do not have zero, so it is not possible to illustrate a calculation such as $5 - 5 = 0$ on the counting line above. In contrast, we can show $5 - 5 = 0$ on a number line. This difference might appear to be minor, but the presence of 0 on the number line and its absence from the counting line indicates the different conceptual bases:

- Counting lines are based on counting and start at 1
- Number lines are based on measuring and can start at 0

Note that we cannot simply insert another square at the start of the counting line above and label it 0, because in counting, the first object is 1, not 0.

Additionally counting lines cannot show fractions or decimals or, later, negative numbers. Where would we label $8 \frac{1}{2}$ on these objects? If 8 refers to a complete object and 9 to the next, there is nothing that can be labelled $8 \frac{1}{2}$. In contrast, a number line based on distance can show all these numbers, when students are ready.

The process of counting objects produces counting numbers.
Measuring produces other types of numbers.
Number lines come from measurement.
The diagram below shows a (horizontal) number line with 0 at the start (on the left). As well as showing the same calculation as above (9 + 5 = 14) by starting at the point labelled 9 and moving a distance of 5 to the point labelled 14, we can also show clearly 5 – 5 = 0. The interpretation is different: I start 9 units away from the origin and move 5 further away, to reach 14. Starting 5 units from the origin and moving 5 back towards it brings me exactly back to the origin.

Because it is a model based on distance, the number line represents a simple scale, and the whole numbers must be shown equally spaced from each other. It is important to include zero as the distance from zero represents the size of the number.

Mitchell and Horne (2008) draw attention to a common mistake made by young students, where they count the markers rather than the intervals. In the diagram below, these students would start at 2, counting the markers as 1, 2, 3, 4, so they would say that the length of the red bar was four. The source of this is confusion between the “counting line” and the “number line”. If 2, 3, 4, and 5 label objects, then indeed there are 4 of them indicated by the red line. In contrast, if I start at 2 units away from the origin and then move to 5 units away from the origin, I will have moved 3 units.

Many students make a closely related error when using rulers to measure. They frequently place the end of the object at 1 rather than zero, and then look at the numbers on the markers rather than the intervals. It is critical that students understand how to move around the number line using distances before they use the number line for number operations.

Lehrer (2003) notes that only a minority of young children understand that any point on a scale can serve as a starting point provided they count the unit intervals corresponding to the length of the object they are measuring, and that even older children if working with a non-zero origin will simply read off whatever number on the ruler corresponds with the end point of the object they are measuring. In a longitudinal study of children in years 1 to 3, Lehrer, Jenkins and Osana (1998) presented children with a task similar to the one in the diagram above. A fifth of children focused on the endpoint of the object, (e.g. stating the length above as 5). Another two fifths of students used 1 as the “zero point”, stating the length as 4 units. This use of 1 as the zero point was equally common amongst students in grades 1, 2 and 3, so this certainly needs teaching attention.

**Addition and subtraction using the number line**

There are many methods for using the number line for adding and subtracting whole numbers. Number lines are also useful for illustrating repeated addition as a precursor to
multiplication and repeated subtraction as a precursor to division. The number line enables different methods to be demonstrated, visualised and hence become an object of discussion in the classroom. Some methods for addition are illustrated using the addition $3 + 5$.

- “Counting all” method: staring at zero, the student jumps to 1, 2, 3, then one by one jumps on 5 more to 8.

- “Counting on” method: the student starts at 3 and counts on 5 more.

- Facile number facts: the student starts at 0, and adds 3 and 5. The student also knows that 5+3 will give the same answer. The number line diagram illustrates this.

Ernest (1985) notes that various mistakes that students make when they count the starting number, or are confused whether to count tick-marks or intervals. For example, students may start at 3, and then count the markers 3, 4, 5, 6, 7 rather than the intervals. Some students may start counting 3 at zero and count the markers on the number line, ending on 7 instead of 8. Ernest cautions that when number lines are used to assess students’ understanding of whole number addition, incorrect responses may mean either a lack of understanding of concepts of whole number addition or a lack of knowledge or understanding of the number line model or lack of understanding of the correspondence between them. Similar cautions apply for subtraction.

The empty number line

An empty number line is a schematic aid to mental calculation or to illustrating calculations. On an empty number line, students mark only the numbers they need for their calculation. For example, to add 25 to 38, children can draw a line (see below) showing 38, and use it to record the partial sums when 20 is added (to give 58) and then the next 5 (to give 63). Students can use this as a tool to keep a record of their work, to help with mental calculation, and also to explain their thinking to their teachers and fellow students. Gravemeijer (1994) has investigated how it can be used to illustrate and support addition and subtraction of 2 digit numbers and beyond. Gravemeijer notes that earlier attempts in the 1970s with empty
number lines were not successful as students associated number lines with a ruler which had fixed distances. Students were reluctant, for example, to place numbers such as 379 and 412 ‘somewhere’ on an empty number line with 400 somewhere between them. While a real number line has all distances exactly to scale, a sketch of a mental number line can serve its recording purpose without exactness of position.

In another study, Klein, Beishuizen and Treffers (1998) found the empty number line (really an “empty counting line”) to be a very powerful model for the learning of addition and subtraction up to 100. Grade 2 students first worked with bead strings of 100 beads, each group of 10 beads coloured either red or white. The number line was then introduced as a model of the bead string. Klein, Beishuizen and Treffers note that “by using the empty number line, children can extend their counting strategies and raise the sophistication level of their strategies from counting by ones to counting by tens to counting by multiples of ten” (p. 446):

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38 + 25 =
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![Number line diagram](image)
References